Saarland University
Department 6.2 - Computer Science
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Theoretical Computer Science - WS13/14

Exercise Sheet 7 (due: 19.03.14, 36 points)

Exercise 1: (elementary arithmetic) (9 points)

Let $\mathscr{S}$ be a proof system for elementary arithmetic introduced in the lecture. Prove the following statement:

$$\mathscr{S} \text{ is inconsistent } \iff \forall w \in \mathscr{S}.\, \mathscr{S} \vdash w$$

You may assume the following axioms and proof rules to be present in $\mathscr{S}$:

1. (a) $A \rightarrow (B \rightarrow A)$
   (b) $(A \rightarrow B) \rightarrow ((A \rightarrow (B \rightarrow C)) \rightarrow (A \rightarrow C))$
2. $A \mid A \rightarrow B \mid B$ (modus ponens)
3. $A \rightarrow (B \rightarrow A \wedge B)$
4. (a) $A \wedge B \rightarrow A$
   (b) $A \wedge B \rightarrow B$
5. (a) $A \rightarrow A \vee B$
   (b) $B \rightarrow A \vee B$
6. $(A \rightarrow C) \rightarrow ((B \rightarrow C) \rightarrow (A \vee B \rightarrow C))$
7. $(A \rightarrow B) \rightarrow ((A \rightarrow \neg B) \rightarrow \neg A)$
8. $\neg\neg A \rightarrow A$

Exercise 2: (constructible functions) (6 + 6 points)

Prove that

1. function $s(n) = n$ is space constructible,
2. function $t(n) = n^2$ is time constructible.

Exercise 3: (space hierarchy) (6 + 6 points)

The theory of space hierarchy says, that one can solve more problems with asymptotically more space.

1. Show the following lemma:

   For every total computable function $s : \mathbb{N}_0 \rightarrow \mathbb{N}_0$ there exists a space constructible, strictly monotonically increasing function s', s.t., $s'(n) > s(n)$ for all n.

2. Use 1) and the space hierarchy theorem to prove the following lemma:

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For every total computable function $s : \mathbb{N}_0 \rightarrow \mathbb{N}_0$, there exists a decidable language, which is not accepted by any $\mathcal{O}(s(n))$-space bounded TM.

Exercise 4: (abstract complexity theory) (1 + 2 points)

In the lecture we have introduced the time complexity $\psi_u(n)$ and the tape complexity $\beta_u(n)$ of the TM M_u. Show that the following sets are decidable:

1. $\{bin(u)\#bin(n)\#bin(x) \mid u, n, x \in \mathbb{N}_0 \wedge \psi_u(n) = x\}$
2. $\{bin(u)\#bin(n)\#bin(x) \mid u, n, x \in \mathbb{N}_0 \wedge \beta_u(n) = x\}$