Saarland University

Department 6.2 - Computer Science
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Theoretical Computer Science - WS13/14

Exercise Sheet 5 (due: 12.03.14, 39 points + 5 bonus points)

Exercise 1: (Church's thesis) (5 bonus + 3 + 3 points)

Church's thesis claims that every Turing-computable function $f : \mathbb{N}_0^r \to N$ is a μ-recursive function. In the lecture we defined the function $\Psi : (A \cup Z)^+ \to \mathbb{N}_0$, which encodes a given configuration of the Turing machine as a natural number. Let $p = \#(A \cup Z)$ and $P = p + 1$, then it holds

$$\forall a \in A \cup Z.\ \Psi(a) \neq 0, \text{ injektiv}$$
$$\Psi(b_1, \ldots, b_{|b|}) \equiv \sum_{j=1}^{|b|} \Psi(b_j) \cdot P^{|b|-j}$$

We have also introduced a number of μ-recursive functions, which satisfy the following conditions:

$$\tilde{\Delta}(\Psi(k)) = \Psi(B\Delta(k)B)$$
$$End(\Psi(k)) = \begin{cases} 1 & k \text{ is the end configuration} \\ 0 & \text{otherwise} \end{cases}$$
$$D(0, x) = x$$
$$D(n+1, x) = \tilde{\Delta}(D(n, x))$$
$$T = \mu(g), \quad \text{where } g(i, x) = End(D(i, x))$$
$$E(x_1, \ldots, x_r) = \Psi(Bz_0 bin(x_1)\#bin(x_2)\# \ldots \#bin(x_r)B)$$
$$Q(\Psi(B \ldots Bz_n bin(x)B \ldots B)) = x.$$

With the help of these functions, one constructs the function f:

$$f(x_1, \ldots, x_r) = Q(D(T(E(x_1, \ldots, x_r)), E(x_1, \ldots, x_r))).$$

Hence, the function f is μ-recursive. To define all the functions listed above we used a number of auxiliary functions, such as $concat(\Psi(a), \Psi(b))$, $prefix(\Psi(b), i)$, etc.

1. We defined the function $prefix(x, i)$ as

   $$prefix(x, i) \equiv \lfloor x/(p+1)^{L(x)-i} \rfloor,$$

   where $L(x) = |x|$. Prove that

   $$prefix(\Psi(b), i) = \Psi(b_1, \ldots, b_i).$$

2. Give a formal definition of the function $End : \mathbb{N} \to \mathbb{B}$, which satisfies

   $$End(\Psi(k)) = \begin{cases} 1 & k \text{ is the end configuration} \\ 0 & \text{otherwise.} \end{cases}$$

3. Give a formal definition of the function $Q : \mathbb{N} \to \mathbb{B}$, which satisfies

   $$Q(\Psi(B \ldots Bz_n bin(x)B \ldots B)) = x.$$

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Exercise 2: (Turing machine) (4 points)

Define a 1-tape (regular) Turing machine $M = (A, Z, \delta, z_s, E)$ which shifts the content of the tape one symbol to the right. You can assume $A = \{0, 1, B\}$ and the tape initially to contain at least one non-empty symbol.

Exercise 3: (halting problem) (3 points)

In the lecture we defined the special halting problem K and the halting problem H as

$$K = \{u \mid M_u \text{ started with } u \text{ halts}\}$$
$$H = \{u\#v \mid M_u \text{ started with } v \text{ halts}\}.$$

A language L is reducible to the language L' (i.e., $L \leq L'$) iff there exists a total computable function f such that

$$\forall w : w \in L \leftrightarrow f(w) \in L'.$$

Show that the special halting problem is reducible to the halting problem:

$$K \leq H.$$

Exercise 4: (reducible languages) (2×6 points)

A language L is decidable iff its characteristic function X_L is Turing-computable. Prove or disprove the following statements:

1. for all L, L', if L' is decidable and $L \leq L'$ then L is decidable,
2. for all L, L', if L is not decidable and $L \leq L'$, then L' is not decidable.
3. for all L, L', if L is decidable and $L \leq L'$, then L' is decidable,
4. for all L, L', L'', if $L \leq L'$ and $L' \leq L''$, then $L \leq L''$ (transitivity),
5. for all L, L', $L \leq L'$ (reflexivity),
6. for all L, L', if $L \leq L'$ and $L' \leq L$, then $L = L'$ (antisymmetry),

You may assume $L, L'' \subset \{0, 1, \#\}^+$.

Exercise 5: (RE languages) (2×7 points)

A language L is called recursively enumerable (RE) iff there exists a Turing machine M, s.t., started with the blank tapes successively prints all elements of L on tape 1. Prove or disprove the following statements:

1. for all L, if L as decidable and L is not RE, then L is regular,
2. for all L, L', if $L \leq L'$ and L' is RE, then L is RE,

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3. for all L, L', if $L \leq L'$ and L is not RE, then L' is not RE,
4. for all L, if L is context free, then L is RE,
5. for all L, if L is RE, then $\overline{L}$ is RE,
6. for all L, if $L = \{u \in \{0,1\}^* \mid L(M_u) \text{ is context free}\}$,
7. for all L, if L is context free, then L is decidable,

You may assume $L, L'' \subset \{0, 1, \#\}^+$.