Saarland University
Department 6.2 - Computer Science
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Theoretical Computer Science - WS13/14

Exercise Sheet 3 (due: 5.03.14, 50 points + 6 bonus points)

Organizational notes:

- On Monday 3.03.14 (Rosenmontag) there is no lecture, no tutorial, and no new exercise sheet!
- This exercise sheet is due on Wednesday, 5.03.14.
- This exercise sheet contains some *bonus* points. This means you can get more than 100% of points for this exercise sheet.

Exercise 1: (Chomsky normal form and CYK algorithm) (5 + 5 points)

Consider the context free grammar $G = (T, N, P, S)$, where $T = \{a, b\}$, $N = \{S, X\}$ and

$$P = \{S \to Xb, X \to aXa \mid aXb \mid bXa \mid bXb \mid a\}.$$

- A context free grammar is in Chomsky normal form if all its productions $\alpha \to \beta$ are of the form

  $$\begin{aligned}(\alpha = S \wedge \beta = \epsilon) \vee \\ (\alpha \in N \wedge \beta \in T) \vee \\ (\alpha \in N \wedge \beta \in NN).\end{aligned}$$

  In the lecture an algorithm to convert a grammar to Chomsky normal form was provided. The algorithm has the following steps:

  1. replace in G all terminals by dedicated non-terminals
  2. eliminate ϵ productions (Note: not relevant for the given grammar)
  3. eliminate chain rules
  4. eliminate productions with more than two non-terminals at the right-hand side

  Run this algorithm on grammar G and construct a Chomsky normal form grammar G', s.t. $L(G) = L(G')$.

- Run the Cocke - Younger - Kasami (CYK) Algorithm to check if $ababab \in L(G')$. Compute all $S'_{i,j}s$ needed to provide an answer.

Exercise 2: (arithmetic theorems) (3 + 3 + 3 + 3 points)

Prove the following theorems using definitions and the already proven theorems given in the lecture. Alternatively you can lookup the same definitions and theorems in the "'System architecture"' textbook (the link to the PDF can be found in the Bibliography section on the website of the lecture). For every step of the proof state what definition or theorem you are using.

1. $0 \cdot x = 0$
2. $(x + y) \cdot z = x \cdot z + y \cdot z$

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3. $x \cdot y = y \cdot x$

4. $8 + 3 = 11$

Hint: prove statements (1) - (3) by induction.

Exercise 3: (primitive recursive functions) (3 + 3 + 3 + 3 + 3 points)
In the lecture we defined 3 base cases of primitive recursive (PR) functions (i.e., constant functions, successor functions and production functions) and 2 rules for defining new PR functions (i.e, a composition rule and a primitive recursion rule). In this exercise you need to formally define the following PR functions:

1. $+(x, y)$, s.t., $+(x, y) = x + y$

2. $sg(x)$, s.t. $sg(x) = 0$ if $x = 0$ and $sg(x) = 1$ otherwise,

3. $P(x)$, s.t. $P(x) = x - 1$ if $x \geq 1$ and $P(x) = 0$ otherwise,

4. $-(x, y)$, s.t. $-(x, y) = x - y$ if $x \geq y$ and $-(x, y) = 0$ otherwise,

5. $\cdot(x, y)$, s.t. $\cdot(x, y) = x \cdot y$.

Exercise 4: (uncountable sets) (4 points + 6 bonus points)
A set A is countable iff there exists bijective function f, s.t., $f : \mathbb{N}_0 \to A$.

1. Show that the set of all total functions $g : \mathbb{N}_0 \to \{0, 1\}$ is not countable.

2. Show that the set $2^{\mathbb{N}_0}$, i.e., the power set of the set of natural numbers, is not countable.

Exercise 5: (countable sets) (4 points)
Let $A = \{a, b\}$. Show that A^* is countable.

Exercise 6: (PR functions) (5 points)
For a function $f \in \mathbb{N}_0^r \to \mathbb{N}_0$, let there be finitely many x, s.t., $f(x) \neq 0$. Show that f is primitive recursive.