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Computer Architecture I – WS 06/07

Solution to Sheet 7 Exercise 3 **(13.01.07)**

To show:

1. $sI(k, t+1) = sI(k, t) +$ if ue_k^t then 1 else 0
2. $sI(k, t) = sI(k+1, t) +$ if $full_{k+1}^t$ then 1 else 0

Proof of 1.: Case split on ue_k^t

- case 1: $ue_k^t = 0$

  easy, by definition of scheduling function

- case 2: $ue_k^t = 1$

$$\begin{array}{rcll} sI(k, t+1) & = & sI(k-1, t) & \text{by scheduling function} \\ & = & sI(k, t) + 1 & \text{with second part and } ue_k^t \Rightarrow full_k^t \end{array}$$

Proof of 2.: Case split on $full_{k+1}^t$

- case 1: $full_{k+1}^t = 0$

  Because $(full_{k+1}^t = 0) \Rightarrow (ue_{k+1}^t = 0)$, easy with scheduling function

- case 2: $full_{k+1}^t = 1$

  Because $full_{k+1}^t \Leftrightarrow ue_k^{t-1} \vee stall_{k+1}^{t-1}$, we show

$$ue_k^{t-1} \vee stall_{k+1}^{t-1} \Leftrightarrow sI(k, t) = sI(k+1, t) + 1$$

  In the following, we will use the auxiliary lemma that for any cycle t the values of the scheduling functions of two adjacent stages are either equal or the value of the smaller stage is greater by one. This lemma should be clear and is shown in very similar proof like the one below. It is not to hard to show but lengthy. Therefore, we omit it here.

  Moreover, we will have to make the following convention:

$$\neg full_k^t \Rightarrow \neg stall_k^t$$

  We show this case by an induction on t and a case split on ue_k^{t-1} and ue_{k+1}^{t-1} in the induction step.

  1. $t = 0$ Easy using the fact that

$$full_k^t \Rightarrow t \geq k$$

  2. $t - 1 \rightarrow t$

- $ue_k^{t-1} = 1$ and $ue_{k+1}^{t-1} = 1$
  to show:

$$\begin{array}{rrcll} & sI(k,t) & = & sI(k+1,t)+1 & \\ \Leftrightarrow & sI(k-1,t-1) & = & sI(k,t-1)+1 & \text{by scheduling function} \\ \Leftrightarrow & sI(k,t-1)+1 & = & sI(k,t-1)+1 & \text{by I.H. for } t-1 \text{ and } k-1 \end{array}$$

  The I.H. can be applied in that way because $ue_k^{t-1} \Rightarrow full_k^{t-1}$.

- $ue_k^{t-1} = 1$ and $ue_{k+1}^{t-1} = 0$
  since the left side of the equivalence is true, we have to show that

$$sI(k,t) = sI(k+1,t) + 1$$

$$\begin{array}{rrcll} & sI(k,t) & = & sI(k+1,t)+1 & \\ \Leftrightarrow & sI(k-1,t-1) & = & sI(k+1,t)+1 & \text{by scheduling function} \\ \Leftrightarrow & sI(k,t-1)+1 & = & sI(k+1,t)+1 & \text{by I.H. and } ue_k^{t-1} \Rightarrow full_k^{t-1} \\ \Leftrightarrow & sI(k,t-1) & = & sI(k+1,t) & \\ \Leftrightarrow & sI(k,t-1) & = & sI(k+1,t-1) & \text{by scheduling function} \end{array}$$

  Assume the equation does not hold, then holds by our auxiliary lemma that

$$sI(k,t-1) = sI(k+1,t-1) + 1$$

  which would imply that $full_{k+1}^{t-1}$ holds by the I.H, but this is a contradiction to the first lemma showed in this exercise ($ue_k^{t-1} \wedge full_{k+1}^{t-1} \Rightarrow ue_{k+1}^{t-1}$).

- $ue_k^{t-1} = 0$ and $ue_{k+1}^{t-1} = 1$
  we have to show that

$$\begin{array}{rrcll} & stall_{k+1}^{t-1} & \Leftrightarrow & sI(k,t) = sI(k+1,t)+1 & \\ \Leftrightarrow & stall_{k+1}^{t-1} & \Leftrightarrow & sI(k,t-1) = sI(k,t-1)+1 & \text{by scheduling function} \end{array}$$

  Hence, the right side of the equivalence is always false and we have to show that $stall_{k+1}^{t-1}$ does not hold which follows from the definition of the ue_{k+1}^{t-1} signal.

- $ue_k^{t-1} = 0$ and $ue_{k+1}^{t-1} = 0$
  we again have to show that

$$\begin{array}{rrcll} & stall_{k+1}^{t-1} & \Leftrightarrow & sI(k,t) = sI(k+1,t)+1 & \\ \Leftrightarrow & stall_{k+1}^{t-1} & \Leftrightarrow & sI(k,t-1) = sI(k+1,t-1)+1 & \text{by scheduling function} \end{array}$$

  Using the I.H. for $t-1$:

$$sI(k,t-1) = sI(k+1,t-1) + 1 \Leftrightarrow full_{k+1}^{t-1}$$

  we show that

$$stall_{k+1}^{t-1} \Leftrightarrow full_{k+1}^{t-1}$$

  From the definition of ue_{k+1}^{t-1} we have that $full_{k+1}^{t-1}$ implies $stall_{k+1}^{t-1}$, if $ue_{k+1}^{t-1} = 0$. The other direction is given by the convention we made.