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Department 6.2 - Computer Science
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Theoretical Computer Science - WS13/14

Exercise Sheet 6 (due: 17.03.14, 36 points)

Exercise 1: (Rice's theorem) (6 points)
Let $R = \{f : A^* \to A^*\}$ be the set of all Turing-computable functions and $R_1 \subset R$ be a non-empty set which contains functions satisfying some property. Further, let $R' = \{u \mid f_{M_u} \in R_1\}$ be the set of (gödelizations of) Turing machines which compute the functions from R_1. The theorem of Rice says, that set R' is undecidable. In the proof given in the lecture we did a case split on the function Ω, which is undefined for any input. We considered the case when $\Omega \in R_1$. Here, you need to provide the proof for the case $\Omega \notin R_1$.

Exercise 2: (recursion theorem) (5 points)
As a consequence of the recursion theorem given in the lecture, for any Turing complete programming language one can write a program that prints itself. Write such a program in a language you like. Comment your program so that anyone can understand what it does.

Exercise 3: (proof system) (3 + 3 points)
Let $\mathscr{S} = (\Sigma, L, A, S)$ be a proof system. Prove the following statements:

1. the set $L' = \{w \mid w \text{ is a proof in } \mathscr{S}\}$ is decidable,
2. the set $L' = \{w \mid S \vdash w\}$ is RE

Exercise 4: (Gödel's 1) (3 points)
In the proof of the first Gödel's incompleteness theorem we have introduced a function $\Psi : A \cup Z \cup \{|\} \to \{1, \ldots, p-1\}$, where p is a **prime** number. Explain why we have chosen the number p to be prime.

Exercise 5: (Gödel's 1) (3 + 3 points)
Define arithmetic predicates which have the following meaning:

1. i is a prime number,
2. there are infinitely many twin primes[1],

Exercise 6: (Gödel's 1) (5 points)
Let M be a 1-tape Turing Machine, v be its input and $t \in \mathbb{N}_0$ be some constant. Define an arithmetic predicate, which is true if and only if Machine M halts in at most t steps. You may use the definitions given in the lecture.

Exercise 7: (elementary arithmetic) (5 points)
Let $\mathscr{S} = (\Sigma_E, L_E, A_E, S_E)$. Prove or disprove: the set $L' = \{w \mid \mathscr{S} \vdash w\}$ is undecidable

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[1] A twin prime is a prime number which differs from another prime number by 2